1st-order initial-value problems

Consider the following IVP:

$$y^{(1)}(t) = -3ty(t)$$
$$y(0) = 2$$

1. Approximate the solution to $y(0.4)$ using four steps of Euler's method.

$y_0 = 2$
$y_1 = 2$
$y_2 = 1.94$
$y_3 = 1.8236$
$y_4 = 1.659476$

2. Approximate the solution to $y(0.4)$ using two steps of Heun's method.

$y_0 = 2$
$y_1 = 1.88$
$y_2 = 1.568672$

3. Approximate the solution to $y(0.4)$ using one step of the 4th-order Runge-Kutta method.

$y_0 = 2$
$y_1 = 1.572992$

4. The actual solution is $y(t) = 2e^{-\frac{3}{2}t^2}$ and $y(0.4) = 1.573255722133107$ to sixteen significant digits. What is the absolute error of each of the above three approximations, and what would you expect the error to be if you were to double the number of steps (eight steps of Euler with $h = 0.05$, four steps of Heun with $h = 0.1$, and two steps of the 4th-order Runge-Kutta method with $h = 0.2$)?

Answers: The absolute errors are 0.08622, 0.004584 and 0.0002637, so if we were to halve the step size with each algorithm, the next errors should be approximately half, one quarter, and one sixteenth of these, respectively, or 0.04311, 0.001146 and 0.00001648, respectively.

5. Do the above three using half the step size to approximate $y(0.4)$ and see if the error drops as expected.

Answer (displayed to nine significant digits):

2, 1.985, 1.955225, 1.91123244, 1.85389546, 1.78437438, 1.70407754, 1.61461347 (0.04136)
2, 1.97, 1.883123, 1.74697321, 1.57297468 (0.0002810)
2, 1.883528, 1.57324856 (0.000007166)

The error for Euler's method dropped closes to what was expected; however, the errors for Heun and the 4th-order Runge-Kutta dropped more, likely because of the small number of steps.

6. Why does the above analysis use four steps with Euler, two steps with Heun and one step with 4^{th}-order Runge-Kutta? Why not use $h = 0.1$ and four steps with each?

Answer: In all likelihood, the most expensive operation is not the calculation required for the methods, but the function calls. Thus, we want to evaluate each with the same number of function evaluations. Euler's method uses one function call per step, Huen's uses two, and the 4^{th}-order Runge-Kutta method uses four, and thus, we should use the same number of function evaluations.

Consider the following IVP:

$$y^{(1)}(t) = -3\cos(ty(t))$$
$$y(0) = 2$$

This does not have a solution that can be expressed in terms of known functions.

7. Approximate the solution to $y(0.4)$ using four steps of Euler's method.

$y_0 = 2$
$y_1 = 1.7$
$y_2 = 1.404324569927132$
$y_3 = 1.116079753425977$
$y_4 = 0.832739298619968$

8. Approximate the solution to $y(0.4)$ using two steps of Heun's method.

$y_0 = 2$
$y_1 = 1.417173600341496$
$y_2 = 0.859225940384325$

9. Approximate the solution to $y(0.4)$ using one step of the 4th-order Runge-Kutta method.

$y_0 = 2$
$y_1 = 0.843080094023240$

10. Can you trust this solution?

Answer: Yes, as the error analysis and the previous examples give confidence that the solutions are correct. Also, not that for small values of t, we have $\cos(ty(t)) \approx 1$, so the IVP is approximately:

$$y^{(1)}(t) = -3$$
$$y(0) = 2$$

The solution to this IVP is $y(t) = 2 - 3t$, and for this IVP, $y(0.4) = 0.8$, which is very close to the approximations we see above.

11. Suppose you solved an IVP with *n* steps and then with 2*n* steps with the three above algorithms, and the differences between these approximations were 0.1 in each case. What is a good estimator of the error of the more accurate approximation?

Answer: If this was for Euler method, the error for the 2*n* approximation would be 0.1; if it was Heun's method, the error of the 2*n* approximation would be 0.1/3 or 0.03333; and if this was the 4^{th}-order Runge-Kutta method, the error of the 2*n* approximation would be approximately 0.1/15 or 0.006666.